Grassmannians the easy way

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Let V be a finite-dimensional $\mathbb{R}$-vector space, and set $n := \dim V$. For any $k \in \{1, ..., n-1\}$, $G_k(V)$ is the set of all k-dimensional subspaces of V.

In Example 1.36 of John M. Lee's *Introduction to Smooth Manifolds*, the author presents a detailed example describing a smooth manifold structure on $G_k(V)$, using the Smooth Manifold Chart Lemma. Here we clarify how the charts are defined, and show that they are smoothly compatible. The charts will be defined as bijections between subsets of $G_k(V)$ and all of $M(k \times (n-k), \mathbb{R})$, which for our purposes is close enough to $\mathbb{R}^{k(n-k)}$ to satisfy the definition of a manifold (though of course it is trivial to make such charts into honest-to-goodness maps to $\mathbb{R}^{k(n-k)}$, e.g. by concatenating the rows of a matrix to make a single row).

Central to the construction is the following elementary set-theoretic fact about functions $X \to Y$ viewed as a particular kind of relation, i.e. as a particular kind of subset of $X \times Y$. The proof is left to the reader.

Lemma 1. *Let X and Y be sets, and let $\pi_X : X \times Y \to X$ be the canonical projection associated with the direct product. Then a subset $R \subseteq X \times Y$ represents a function $X \to Y$ iff $\pi_X|_R$ is a bijection.* □

Products (and their projections) are ubiquitious in mathematics. In the category of vector spaces, the property that the restriction of a projection to a subobject is a bijection (equivalently, an isomorphism) is related to another important property, on which the smooth structure of $G_k(V)$ is based. Before we explain this, however, we first make a comment about direct sums and direct products of vector spaces. These are not the same thing, but the distinction between finite direct products and finite direct sums is subtle, and while there are some situations in which that distinction is important and/or useful, it has no bearing on what we are discussing here. Thus we will make no distinction between the direct product $V \times W$ and the direct sum $V \oplus W$ of two vector spaces.

Definition. *Let V be a vector space and $Q \subseteq V$ a subspace. A complement of Q in V is a subspace $S \subseteq V$ with the property that V is the internal direct sum of S and Q. That is, $S \cap Q = 0$, and $S + Q = V$.*

Lemma 2. *Let V be a vector space, and suppose V is an internal direct sum $P \oplus Q$. Let $\pi_P : V \to P$ be the canonical projection; then for any subspace $S \subseteq V$, $S \cap Q = 0$ iff $\pi_P|_S$ is injective, and $S + Q = V$ iff $\pi_P|_S$ is surjective.*

Proof. Ker $\pi_P|_S = S \cap \text{Ker } \pi_P = S \cap Q$, so $S \cap Q = 0$ iff $\pi_P|_S$ is injective. If $\pi_P|_S$ is surjective, then for any $p \in P$ there is $q \in Q$ with $p + q \in S$; it follows that $P \subseteq S + Q$ and hence $S + Q = V$. Conversely if $S + Q = V$ then for any $p \in P$ there are $s \in S$ and $q \in Q$ with $s + q = p$; thus $\pi_P s = \pi_P(p - q) = p$, and therefore $\pi_P|_S$ is surjective. □

Corollary. *$S \subseteq V$ is a complement of Q iff $\pi_P|_S$ is an isomorphism.* □

We now have two ways of thinking about subsets $S \subseteq P \times Q$ having the property that $\pi_P|_S$ is a bijection. Lemma 1 relates such subsets to functions $P \to Q$, while Lemma 2 relates subspaces with this property to complements of Q. As it turns out, it we restrict our attention to subspaces in Lemma 1, we can "connect" these two relations. First, we have the following result, whose proof is left to the reader.

Lemma 3. *In Lemma 1, if $P \times Q$ is a direct product of vector spaces, then the subset S is a subspace iff the corresponding function $P \to Q$ is linear.* □

This leads us to the main result we need:

Proposition. *Let V be a vector space, and suppose V is an internal direct sum $P \oplus Q$. Then there is a bijection between the set of complements of Q in V and the set $L(P, Q)$ of linear maps $P \to Q$.*

Proof. Using the above results, we immediately get the following equivalences for a subset $S \subseteq V$:

$$\begin{aligned} S \text{ is a complement of } Q &\iff S \text{ is a subspace and } \pi_P|_S \text{ is an isomorphism} \\ &\iff S \text{ is a subspace and } \pi_P|_S \text{ is a bijection} \\ &\iff S \text{ is a subspace, and represents a function } P \to Q \\ &\iff S \text{ represents a linear function } P \to Q. \end{aligned}$$

□

Remark. Note that none of the above results require that P or Q be finite-dimensional; moreover they are valid for any base field, and in fact (by the same proof) for any module over an arbitrary ring (replacing "subspace" with "submodule" everywhere).

To define a chart on $G_k(V)$, we start by fixing a decomposition $P \oplus Q$ of V with dim $P = k$, along with specific bases $E_P := \{e_1, ..., e_k\}$ of P and E_Q of Q (we don't need to explicitly name the elements of E_Q). The domain of our chart is

$$U_Q := \{S \subseteq V : S \text{ is a complement of } Q\}.$$

By the Corollary, any such S has dimension k; thus $U_Q \subseteq G_k(V)$. The chart φ determined by P, Q, E_P and E_Q is the composite of the map $\gamma : U_Q \to L(P, Q)$ given by the Proposition, and the (linear) map $R : L(P, Q) \to M((n-k) \times k, \mathbb{R})$ taking $\sigma \in L(P, Q)$ to its (E_P, E_Q)-matrix representation. Note that γ depends only on the decomposition $P \oplus Q$, while R depends on the choice of bases E_P and E_Q. Since γ and R are both bijections, the image of φ is all of $M((n-k)\times k, \mathbb{R})$, so (i) of Lemma 1.35 is satisfied.

In order to compute the transition maps between a pair of such charts, we need to know how to go in the opposite direction. This is done as follows: given $M \in M(k \times (n-k), \mathbb{R})$, let $\sigma : P \to Q$ be the corresponding linear map (i.e. $\sigma := R^{-1}M$). By definition, the ith column of M is the E_Q-representation of σe_i. Thus the ith column of the $n \times k$ matrix

$$\begin{bmatrix} I_k \\ M \end{bmatrix}$$

represents $(e_i, \sigma e_i)$. These are all elements of the corresponding subspace S; moreover, there are k of them and they are clearly linearly independent, so they form a basis of S (this is essentially the content of Problem 1-10 in Lee's book).

Now consider another chart on $G_k(V)$, defined by a decomposition $P' \oplus Q'$ of V with dim $P' = k$, and bases $E_{P'} := \{e'_1, ..., e'_k\}$ for P' and $E_{Q'}$ for Q'. To satisfy part (ii) of Lemma 1.35, we must show that $\varphi(U_Q \cap U_{Q'})$ is an open subset of $M((n-k)\times k, \mathbb{R})$. Fix $S \in U_Q$, then $S \in U_{Q'}$ iff S is also a complement of Q'. By Lemma 2, this is true iff $\pi_{P'}|_S$ is an isomorphism. Since dim $P' = k =$ dim S, this is true iff $\pi_{P'}|_S$ is surjective, that is if $\pi_{P'}S = P'$. Setting $M := \varphi S$, we know the columns of the matrix $\left[\begin{smallmatrix} I_k \\ M \end{smallmatrix}\right]$ are an E-basis for S. To compute $\pi_{P'}S$, let F be the change-of-basis matrix between E and E'; this is a constant matrix (i.e. it depends only on the bases E and E'). Then the columns of

$$F \begin{bmatrix} I_k \\ M \end{bmatrix}$$

are an E'-basis for S, and therefore $\pi_{P'}S$ is spanned by the columns of the $k \times k$ matrix

$$G := \begin{bmatrix} I_k & 0 \end{bmatrix} F \begin{bmatrix} I_k \\ M \end{bmatrix}.$$

(Here the 0 in the first matrix is the $k \times (n-k)$ zero matrix.) And the columns of G span P' iff G has full rank, equivalently det $G \neq 0$.

Now the entries of G are polynomials in the entries of M, so the map $M \mapsto G$ is continuous, and therefore the map $M \mapsto \det G$ is also continuous. Then $\varphi(U_Q \cap U_{Q'})$ is the inverse image of the open set $\mathbb{R} - \{0\}$ under this map, so it is open.

Now let ψ be the chart defined by P', Q', $E_{P'}$ and $E_{Q'}$ above; we next show that the transition map $\psi \circ \varphi^{-1}$ is smooth. We start with a matrix $M \in M(k\times(n-k), \mathbb{R})$ whose image lies in $U_Q \cap U_{Q'}$; that is, it is a complement of both Q and Q'. We apply the reverse procedure above to get a basis for S of the form $\{(e_i, \sigma e_i)\}$. We then need to compute the $(E_{P'}, E_{Q'})$-matrix representation of the linear map $\sigma' : P' \to Q'$ corresponding to $S \subseteq P' \oplus Q'$. This is equivalent to transforming the columns of the matrix $\left[\begin{smallmatrix} I_k \\ M \end{smallmatrix}\right]$ representing the vectors $(e_i, \sigma e_i)$ in the basis E into a matrix of the same form whose columns represent a basis for S in the basis E. How do we do this?

Well, we can translate the basis $\{(e_i, \sigma e_i)\} \subseteq P \oplus Q$ into *some* basis in terms of the decomposition $P' \oplus Q'$ simply by applying the change-of-basis matrix F. Thus $F\left[\begin{smallmatrix} I_k \\ M \end{smallmatrix}\right]$ is an $n \times k$ matrix whose columns are an E'-basis for S. To get a basis of the desired form, we perform column operations on this matrix to get the identity matrix at the top, i.e. until our matrix has the form

$$\begin{bmatrix} I_k \\ M' \end{bmatrix}.$$

We know we can do this because S is a complement of Q', and thus it has a basis whose elements have the form $(e'_i, \sigma' e'_i)$ with respect to $P' \oplus Q'$, and the E'-representation of such elements yields a matrix of the above form.

The required column operations are effected by right multiplication by a $k \times k$ invertible matrix H, so after doing this we get

$$F \begin{bmatrix} I_k \\ M \end{bmatrix} H = \begin{bmatrix} I_k \\ M' \end{bmatrix}.$$

Since right multiplication by H yields the identity matrix in the top k rows of this matrix, H is in fact the inverse of the $k \times k$ submatrix at the top of $F\left[\begin{smallmatrix} I_k \\ M \end{smallmatrix}\right]$, which is just the matrix G defined above. That is, $H = G^{-1}$. Now we already know that the entries of G are polynomials in the entries of M, and therefore (by Cramer's rule) the entries of H are rational functions in the entries of M, and finally the entries of

$$\begin{bmatrix} I_k \\ M' \end{bmatrix} = F \begin{bmatrix} I_k \\ M \end{bmatrix} H$$

are likewise rational functions in the entries of M.

Summarizing: we can explicitly write down the value of the transition map on a matrix M as follows:

$$\psi \circ \varphi^{-1}(M) = \begin{bmatrix} 0 & I_{n-k} \end{bmatrix} F \begin{bmatrix} I_k \\ M \end{bmatrix} \left(\begin{bmatrix} I_k & 0 \end{bmatrix} F \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1}.$$

(The leftmost matrix extracts the bottom $n - k$ rows of the rest of the expression.) It follows that the transition map $\psi \circ \varphi^{-1} : M \mapsto M'$ is smooth, so (iii) of Lemma 1.35 is satisfied.

Note: we can relate this computation to Lee's proof by observing that the maps

$$A := \pi_{P'}|_P,\ B := \pi_{Q'}|_P,\ C := \pi_{P'}|_Q \text{ and } D := \pi_{Q'}|_Q$$

are just the components of the "change-of-decomposition" map between the decomposition $P \oplus Q$ and $P' \oplus Q'$; here we have broken these down further into the change-of-basis matrix F between E and E'. That is, if we think of A, B, C and D as matrix representations of these maps with respect to the corresponding bases, then F can be expressed as the following block matrix.

$$F = \begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

And indeed, if we substitute this in the expression for $\psi \circ \varphi^{-1}(M)$ above, we get

$$\begin{aligned}
\psi \circ \varphi^{-1}(M) &= \begin{bmatrix} 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} I_k \\ M \end{bmatrix} \left(\begin{bmatrix} I_k & 0 \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} B & D \end{bmatrix} \begin{bmatrix} I_k \\ M \end{bmatrix} \left(\begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1} \\
&= (B + DM)(A + CM)^{-1},
\end{aligned}$$

which is exactly the expression appearing in Lee's example.